Necessary conditions for linear noncooperative N-player delta differential games on time scales[*]

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Abstract

We present necessary conditions for linear noncooperative N-player delta dynamic games on a generic time scale. Necessary conditions for an open-loop Nash-equilibrium and for a memoryless perfect state Nash-equilibrium are proved.

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1 Introduction

In 1988 Stephan Hilger developed in his PhD thesis [8] the theory of *time scales*. The set $\mathbb{R}$ of real numbers and the set $\mathbb{Z}$ of integers are (trivial) examples of time scales. When a result is proved in a general time scale $\mathbb{T}$, one unifies both continuous and discrete analysis. Moreover, since there are infinitely many other time scales, a much more general result is proved. For this reason, one can say that the two main features of the theory of time scales are *unification* and *extension*.

Differential game theory is a relatively new area of Mathematics, initiated in the fifties of the XX century with the works of Rufus Isaacs, that founds many applications in different fields such as economics, politics, and artificial intelligence. The theory has been studied in the context of classical Mathematics, in discrete or continuous time [1, 11]. We trust that it is also possible (with some advantages) to present (delta) differential games in the generic context of time

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scales. To the best of the authors knowledge, this paper represents the first attempt to provide a delta differential game theory on a generic time scale.

The paper is organized as follows. In Section 2 we review some basic definitions and results from the calculus on time scales. In Section 3, necessary conditions for a weak local minimizer of a Lagrange problem on time scales (Theorem 3) are presented, while Section 4 recalls a result that guarantees the uniqueness of the forward solution for a special initial valued problem on time scales (Theorem 7). In Section 5 we introduce the definition of a N-player delta differential game, the notion of Nash-equilibrium, and two types of information structure in a game: open-loop and memoryless perfect state. The main results of the paper appear then in Section 6, where necessary conditions for a linear open-loop Nash-equilibrium and for a linear memoryless perfect state Nash-equilibrium are proved.

2   Calculus on time scales

We briefly present here the necessary concepts and results from the theory of time scales (cf. [3, 4, 9] and references therein). As usual, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote, respectively, the set of real, integer, and natural numbers.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Thus, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ are examples of times scales. Other examples of times scales can be $h\mathbb{Z}$, for some $h > 0$, $[1, 4] \bigcup \mathbb{N}$, or the Cantor set. We assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$$

if $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$$

if $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$.

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ or $\rho(t) < t$, respectively.

The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T}.$$

For a given instant t, $\mu(t)$ measures the distance of t to its right neighbor.

It is clear that when $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$ and $\mu = 0$. When $\mathbb{T} = \mathbb{Z}$, for any $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu = 1$.

In order to introduce the definition of *delta derivative*, we define the set

$$\mathbb{T}^k := \{t \in \mathbb{T} : t \text{ is nonmaximal or left-dense}\}.$$

Thus, $\mathbb{T}^k$ is obtained from $\mathbb{T}$ by removing its maximal point if this point exists and is left-scattered.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^k$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t (i.e., $U =]t - \delta, t + \delta[\cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

We note that when $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists, i.e. if and only if f is differentiable in the ordinary sense at t. When $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at $t \in \mathbb{Z}$ and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = f(t + 1) - f(t).$$

Hence, for $\mathbb{T} = \mathbb{Z}$ the delta derivative of f, f^Δ, coincides with the usual forward difference Δf.

It is clear that if f is constant, then $f^\Delta = 0$; if $f(t) = kt$ for some constant k, then $f^\Delta = k$.

For delta differentiable functions f and g, the next formulas hold:

$$f^\sigma(t) = f(t) + \mu(t)f^\Delta(t),$$
$$(fg)^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t)$$
$$= f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t),$$

where we abbreviate $f \circ \sigma$ by f^σ.

Delta derivatives of higher order are defined in the standard way: we define the rth-*delta derivative* ($r \in \mathbb{N}$) of f to be the function $f^{\Delta^r} : \mathbb{T}^{k^r} \to \mathbb{R}$, provided $f^{\Delta^{r-1}}$ is delta differentiable on $\mathbb{T}^{k^r} := \left(\mathbb{T}^{k^{r-1}}\right)^k$.

The class of continuous functions on $\mathbb{T}$ is too small for a convenient theory of integration. For our purposes, it is enough to define the notion of integral in the class of rd-continuous functions. For a more general theory of integration on time scales, we refer the reader to [4]. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists (finite) at left-dense points. For $\mathbb{T} = \mathbb{R}$ rd-continuity coincides with continuity.

We denote the set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, and the set of all delta differentiable functions with rd-continuous derivative by $C_{\mathrm{rd}}^1(\mathbb{T}, \mathbb{R})$.

It can be shown that every rd-continuous function f possess an *antiderivative*, i.e. there exists a function F with $F^\Delta = f$, and in this case the *delta integral* is defined by

$$\int_a^b f(t)\Delta t := F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

This integral has the following property:

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)\,.$$

Let $a, b \in \mathbb{T}$ and $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. It is easy to prove that

1. for $\mathbb{T} = \mathbb{R}$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right hand side is the usual Riemann integral;

2. for $\mathbb{T} = \mathbb{Z}$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$ if $a < b$, $\int_a^b f(t)\Delta t = 0$ if $a = b$, and $\int_a^b f(t)\Delta t = -\sum_{t=b}^{a-1} f(t)$ if $a > b$.

Next, we present the integration by parts formulas: if $a, b \in \mathbb{T}$ and $f, g \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, then

1. $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\Delta t$;

2. $\int_a^b f(t)g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$.

Similarly to classical calculus, we say that $f : \mathbb{T} \to \mathbb{R}^n$ is a *rd-continuous* function if each component of f, $f_i : \mathbb{T} \to \mathbb{R}$, is a rd-continuous function. The set of all such functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$. The set $C_{rd}^1(\mathbb{T}, \mathbb{R}^n)$ is defined in the usual way.

3 Lagrange problem on time scales

Let $a, b \in \mathbb{T}$ such that $a < b$. In what follows we denote by $[a, b]$ the set $\{t \in \mathbb{T} : a \leq t \leq b\}$. Consider the following Lagrange problem with delta differential side condition:

$$
\begin{aligned}
J[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t))\Delta t &\longrightarrow \min\,, \\
x^\Delta(t) = \varphi(t, x(t), u(t))\,, \quad t \in \mathbb{T}^k\,, & \\
x(a) = x_a\,, &
\end{aligned}
\tag{1}
$$

where we assume that

- $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\varphi : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $(x, u) \to L(t, x, u)$ and $(x, u) \to \varphi(t, x, u)$ are C^1-functions of x and u for each t;

- $x(\cdot) \in C_{rd}^1(\mathbb{T}, \mathbb{R}^n)$ and $u(\cdot) \in C_{rd}(\mathbb{T}, \mathbb{R}^m)$, $m \leq n$;

- for each control function $u(\cdot)$ there exists a unique forward solution $x(\cdot)$ of the initial value problem $x^\Delta(t) = \varphi(t, x(t), u(t))$, $x(a) = x_a$.[1]

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[1] In the linear case, conditions guaranteing the existence and uniqueness of forward solutions are easy to obtain [2] – cf. Section 4.

We borrow from [7] the definition of admissible pair and the definition of weak local minimizer for problem (1). The reader interested in the calculus of variations on time scales is referred to [5, 6] and references therein.

Definition 1. *The pair $(x_*(\cdot), u_*(\cdot))$ is said to be* admissible *for problem (1) if*

1. $x_*(\cdot) \in C_{rd}^1(\mathbb{T}, \mathbb{R}^n)$ *and* $u_*(\cdot) \in C_{rd}(\mathbb{T}, \mathbb{R}^m)$;

2. $x_*^\Delta(t) = \varphi(t, x_*(t), u_*(t))$ *and* $x_*(a) = x_a$.

In order to introduce the notion of weak local minimizer for problem (1), we define the following norms in $C_{rd}(\mathbb{T}, \mathbb{R}^n)$ and $C_{rd}^1(\mathbb{T}, \mathbb{R}^n)$:

$$||y||_\infty := \sup_{t \in \mathbb{T}} \| y(t) \|, \quad y \in C_{rd}(\mathbb{T}, \mathbb{R}^n),$$

and

$$||z||_{1,\infty} := \sup_{t \in \mathbb{T}^k} \| z(t) \| + \sup_{t \in \mathbb{T}^k} \| z^\Delta(t) \|, \quad z \in C_{rd}^1(\mathbb{T}, \mathbb{R}^n),$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Definition 2. *An admissible pair $(x_*(\cdot), u_*(\cdot))$ is said to be a* weak local minimizer *for problem (1) if there exists $r \in \mathbb{R}^+$ such that*

$$J[x_*(\cdot), u_*(\cdot)] \leq J[x(\cdot), u(\cdot)]$$

for all admissible pairs $(x(\cdot), u(\cdot))$ satisfying

$$\| x - x_* \|_{1,\infty} + \| u - u_* \|_\infty < r.$$

Theorem 3 gives necessary conditions for a pair $(x_*(\cdot), u_*(\cdot))$ to be a weak local minimizer of the Lagrange problem (1).

Theorem 3 ([7]). *If $(x_*(\cdot), u_*(\cdot))$ is a weak local minimizer of problem (1), then there exists a multiplier $\psi_*(\cdot) : \mathbb{T} \to \mathbb{R}^n$ that is delta differentiable on $\mathbb{T}^k$, such that*

$$
\begin{aligned}
x_*^\Delta(t) &= \mathcal{H}_{\psi^\sigma}(t, x_*(t), u_*(t), \psi_*^\sigma(t)), \\
\psi_*^\Delta(t) &= -\mathcal{H}_x(t, x_*(t), u_*(t), \psi_*^\sigma(t)), \\
\mathcal{H}_u(t, x_*(t), u_*(t), \psi_*^\sigma(t)) &= 0, \\
\psi_*(b) &= 0,
\end{aligned}
\tag{2}
$$

for all $t \in \mathbb{T}^k$, where the Hamiltonian function $\mathcal{H}$ is defined by

$$\mathcal{H}(t, x, u, \psi^\sigma) := L(t, x, u) + \psi^\sigma \cdot \varphi(t, x, u). \tag{3}$$

Remark 4. *From definition (3) of the Hamiltonian function $\mathcal{H}$, it follows that the first condition in (2) holds for any admissible pair $(x_*(\cdot), u_*(\cdot))$ of problem (1).*

Remark 5. *For the time scale $\mathbb{T} = \mathbb{R}$, Theorem 3 is a particular case of Pontryagin's Maximum Principle [12].*

4 Linear systems on times scales

Let us consider the following initial value problem on a time scale $\mathbb{T}$:

$$\begin{cases} x^\Delta(t) = Ax(t) + f(t), \\ x(a) = x_a, \end{cases} \tag{4}$$

where A is a constant $n \times n$ matrix, $f : \mathbb{T} \to \mathbb{R}^n$ is a rd-continuous function, $a \in \mathbb{T}$ and $x_a \in \mathbb{R}^n$.

Similar to control theory [2], in dynamic game theory we are interested in forward solutions. The purpose of this section is to present conditions assuring problem (4) to have a unique forward solution.

Proposition 6 ([3]). *The initial value problem*

$$\begin{cases} x^\Delta(t) = Ax(t), \\ x(a) = x_a, \end{cases}$$

where A is a constant $n \times n$ matrix, $a \in \mathbb{T}$ and $x_a \in \mathbb{R}^n$, has a unique forward solution.

The *matrix exponential function* (also known as the *fundamental matrix solution*) at a for the matrix A, is defined as the unique forward solution of the matrix differential equation

$$X^\Delta(t) = AX(t),$$

with the initial condition $X(a) = I$, where I denotes the $n \times n$ identity matrix. Its value at t is denoted by $e_A(t, a)$.

Theorem 7 (cf. [2]). *The initial value problem (4) has a unique forward solution of the form*

$$x(t) = e_A(t, a)x_a + \int_a^t e_A(t, \sigma(s)) f(s) \Delta s.$$

5 N-player delta differential games

The (classical) term "N-player dynamic game" is applied to a group of problems in applied mathematics that possess certain characteristics related with conflict problems. The main ingredients in a N-player dynamic game are the players, the control variables, the state variables, and the cost functionals/functions. The relation between state and control variables is given by a differential/difference equation. Two types of games can be considered: cooperative or noncooperative games. In this paper we shall restrict ourselves to noncooperative games. In a noncooperative game the players act independently in the pursuit of their own best interest, each player desiring to attain the smallest possible cost.

Following Jank [10] (see also [1]), we introduce the notion of a N-player delta differential game (noncooperative dynamic game in the context of time scales).

Definition 8. *Let $N \in \mathbb{N}$. We say that*

$$\Gamma_N = (\mathcal{T}, X, U_i, \mathcal{U}_i, \sigma_i, f, x_a, \eta^i, J^i)_{i=1,2,\ldots,N}$$

is a N-player delta differential game if:

1. *$\mathcal{T}$ is a closed nonempty interval of $\mathbb{T}$ ($\mathcal{T}$ is called the* time horizon*);*

2. *X is a finite dimensional Euclidean space (X is called the* state *or* phase space*);*

3. *U_i is a finite dimensional Euclidean space (U_i is called the* control value space *of the i-th player);*

4. *σ_i is a subset of a set of mappings*

$$\{\gamma^i | \gamma^i : \mathcal{T} \times \mathcal{P}(X) \to U_i\}$$

(γ^i is called a strategy *of the i-th player while σ_i is called the* set of possible strategies *of the i-th player);*

5. *x is a mapping from $\mathcal{T}$ to X (x is called the* state variable*);*

6. *$\eta^i : \mathcal{T} \to \mathcal{P}(X)$ is a mapping with the property*

$$\eta^i(t) \subseteq \{x(s) \mid a \le s \le t\}$$

(η^i is called the information structure *of the i-th player);*

7. *$\mathcal{U}_i = \{\gamma^i(\cdot, \eta^i(\cdot)) \mid \gamma^i \in \sigma_i\}$ ($\mathcal{U}_i$ is called the* control space *or* decision set *of the i-th player);*

8. *$f : \mathcal{T} \times X \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_N \to X$ is a mapping that describes an outer force acting on the system by the delta differential equation*

$$x^\Delta(t) = f(t, x(t), u^1(t), \ldots, u^N(t))$$

with the initial condition $x(a) = x_a \in X$ ($u^i \in \mathcal{U}_i$ is called the control function *or* control variable *of the i-th player);*

9. *J^i is a mapping from $\mathcal{U}_1 \times \cdots \times \mathcal{U}_N$ to $\mathbb{R}$ (J^i is called the* cost functional *of the i-th player).*

Remark 9. *For each $i = 1, 2, \ldots, N$, the controls $u^i(\cdot) = \gamma^i(\cdot, \eta^i(\cdot)) \in \mathcal{U}_i$ are built up from chosen strategies γ^i with a specific information structure η^i.*

Remark 10. *The cost functional J^i of the i-th player, $i \in \{1, 2, \ldots, N\}$, depends on the controls of all the N players.*

Definition 11. *A N-tuple of control functions $u(\cdot) = (u^1(\cdot), \ldots, u^N(\cdot))$ is called an* admissible control *for the system Γ_N if u is a rd-continuous function and there exists a unique forward solution $x \in C^1_{rd}(\mathcal{T}, X)$ of the initial value problem $x^\Delta(t) = f(t, x(t), u^1(t), \ldots, u^N(t))$, $x(a) = x_a$.*

We shall consider the following equilibrium concept in a N-player delta differential game.

Definition 12. *A decision N-tuple $u_* = (u_*^1, \ldots, u_*^N) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_N$ is called a* Nash-equilibrium *if for all $i = 1, 2, \ldots, N$,*

$$J^i(u_*^1, \ldots, u_*^N) \leq J^i(u_*^1, \ldots, u_*^{i-1}, u^i, u_*^{i+1}, \ldots, u_*^N)$$

for all $u^i \in \mathcal{U}_i$.

The meaning of the Nash-equilibrium is the following: if the i-th player unilaterally change the strategy u_*^i, then his cost will increase.

We remark that the information available to the players is an important aspect of the game. In this paper we consider open-loop and memoryless perfect state information structures.

Definition 13. *Let Γ_N be a N-player delta differential game. We say that Γ_N has*

1. *Open-loop (OL) information structure if $\eta^i(t) = \{x_a\}, t \in \mathcal{T}$, for all $i \in \{1, 2, \ldots, N\}$;*

2. *Memoryless perfect state (MPS) information structure if $\eta^i(t) = \{x_a, x(t)\}, t \in \mathcal{T}$, for all $i \in \{1, 2, \ldots, N\}$.*

Therefore, in the open-loop information structure each player know only the initial position x_a, while in the memoryless perfect state information structure each player know the phase state $x(t)$ at each time instant t as well as the initial position x_a.

Definition 14. *We say that a control $u^i(t) := \gamma^i(t, \eta^i(t))$ is: (i) an OL control, if the information structure η^i is OL; (ii) a MPS control, if the information structure η^i is MPS.*

6 Main results

Using Theorem 3, we deduce necessary conditions for OL and MPS Nash-equilibrium of N-player games with linear delta differential equations. We deal with cost delta-integral functionals of the following type:

$$J^i(u^1, \ldots, u^N) = \int_a^b L^i(t, x(t), u^1(t), \ldots, u^N(t)) \Delta t, \qquad (5)$$

where we suppose the following (from now on, $\mathbb{T}$ denotes $\mathbb{T} \cap [a, b]$):

- for each $i = 1, 2, \ldots, N$, $L^i : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_N} \to \mathbb{R}$ is a C^1-function of x and $u = (u^1, \ldots, u^N)$ for each t; $\max\{m_1, \ldots, m_N\} \leq n$;

- for each $i = 1, 2, \ldots, N$, $u^i \in \mathcal{U}_i$ is admissible;

- x is the forward solution of the delta differentiable initial value problem

$$\begin{cases} x^\Delta(t) = Ax(t) + B^1 u^1(t) + \cdots + B^N u^N(t) , \\ x(a) = x_a , \end{cases} \tag{6}$$

where A is a constant $n \times n$ matrix, and for each $i = 1, 2, \ldots, N$, B^i is a constant $n \times m_i$ matrix.

Remark 15. *Since each $u^i \in C_{rd}(\mathbb{T}, \mathbb{R}^{m_i})$, and therefore $B^1 u^1 + \cdots + B^N u^N$ is a rd-continuous function, by Theorem 7 the initial value problem (6) has a unique forward solution.*

Theorem 16 (Necessary conditions for a linear OL Nash-equilibrium). *Let Γ_N be a N-player delta differential game, where the cost functional of the i-th player is given by (5). If the decision N-tuple $(u_*^1, \ldots, u_*^N)$ is an OL Nash-equilibrium of Γ_N, and if x_* is the associated trajectory of the state, then there exist delta differentiable functions $\psi^i : \mathbb{T} \to \mathbb{R}^n$ on $\mathbb{T}^k$, $i = 1, 2, \ldots, N$, such that for*

$$H^i(t, x, u^1, \ldots, u^N, (\psi^i)^\sigma) := L^i(t, x, u^1, \ldots, u^N) + (\psi^i)^\sigma \cdot (Ax + B^1 u^1 + \cdots + B^N u^N)$$

one has:

1. $x_*^\Delta(t) = H^i_{(\psi^i)^\sigma}(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t))$;

2. $x_*(a) = x_a$;

3. $(\psi^i)^\Delta(t) = -H^i_x(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t))$;

4. $\psi^i(b) = 0$;

5. $H^i_{u^i}(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t)) = 0$;

for all $i = 1, 2, \ldots, N$ and $t \in \mathbb{T}^k$.

Proof. Suppose that the decision N-tuple $(u_*^1, \ldots, u_*^N)$ is an OL Nash-equilibrium of Γ_N and x_* is the associated trajectory of the state. For each $i = 1, \ldots, N$ consider the functional

$$\widetilde{J}^i : \mathcal{U}_i \to \mathbb{R}$$

defined by

$$\widetilde{J}^i(u^i) := J^i(u_*^1, \ldots, u_*^{i-1}, u^i, u_*^{i+1}, \ldots, u_*^N).$$

Define also

$$f(x, u^1, \ldots, u^N) := Ax + B^1 u^1 + \cdots + B^N u^N.$$

Since $(u_*^1, \ldots, u_*^N)$ is a Nash-equilibrium, then for all $u^i \in \mathcal{U}_i$

$$\widetilde{J}^i(u_*^i) \leq \widetilde{J}^i(u^i).$$

Therefore, $(x_*(\cdot), u_*^i(\cdot))$ is a weak local minimizer of the problem

$$\widetilde{J}^i(u^i) \longrightarrow \min,$$
$$x^\Delta(t) = \widetilde{f}(x(t), u^i(t)), \tag{7}$$
$$x(a) = x_a,$$

where $\widetilde{f}(x, u^i) := f(x, u_*^1, \ldots, u_*^{i-1}, u^i, u_*^{i+1}, \ldots, u_*^N)$. Define

$$\mathcal{H}^i(t, x, u^i, (\psi^i)^\sigma) := H^i(t, x, u_*^1, \ldots, u_*^{i-1}, u^i, u_*^{i+1}, \ldots, u_*^N, (\psi^i)^\sigma),$$

with H^i the Hamiltonian defined in the statement of the theorem. Applying Theorem 3 to $\mathcal{H}^i$ (note that $\widetilde{f}$ is a C^1-function of x and u^i for each t), we conclude that there exists a function $\psi^i : \mathbb{T} \to \mathbb{R}^n$ that is delta differentiable on $\mathbb{T}^k$, such that $\psi^i(b) = 0$,

$$x_*^\Delta(t) = \mathcal{H}^i_{(\psi^i)^\sigma}(t, x_*(t), u_*^i(t), (\psi^i)^\sigma(t))$$
$$= Ax_*(t) + B^1 u_*^1(t) + \cdots + B^i u_*^i(t) + \cdots + B_N u_*^N(t)$$
$$= H^i_{(\psi^i)^\sigma}(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t)),$$

$$(\psi^i)^\Delta(t) = -\mathcal{H}^i_x(t, x_*(t), u_*^i(t), (\psi^i)^\sigma(t))$$
$$= -H^i_x(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t)),$$

and $\mathcal{H}^i_{u^i}(t, x_*(t), u_*^i(t), (\psi^i)^\sigma(t)) = H^i_{u^i}(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t)) = 0$. $\qquad \square$

Remark 17. *Notice that conditions 1 and 2 of Theorem 16 simply assert that x_* is a solution of the initial value problem (6).*

Theorem 18 (Necessary conditions for a linear MPS Nash-equilibrium). *Let Γ_N be a N-player delta differential game, where the cost functional of the i-th player is given by (5). If the decision N-tuple $(u_*^1(\cdot), \ldots, u_*^N(\cdot))$, given by $(\gamma_*^1(\cdot, x_a, x_*(\cdot)), \ldots, \gamma_*^N(\cdot, x_a, x_*(\cdot)))$, is a MPS Nash-equilibrium of Γ_N and if $x_*(\cdot)$ is the associated trajectory of the state, then there exist delta differentiable functions $\psi^i : \mathbb{T} \to \mathbb{R}^n$ on $\mathbb{T}^k$, $i = 1, 2, \ldots, N$, such that for*

$$H^i(t, x, u^1, \ldots, u^N, (\psi^i)^\sigma) := L^i(t, x, u^1, \ldots, u^N) + (\psi^i)^\sigma \cdot (Ax + B^1 u^1 + \cdots + B^N u^N)$$

one has:

1. $x_*^\Delta(t) = H^i_{(\psi^i)^\sigma}(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t))$;

2. $x_*(a) = x_a$;

3. $(\psi^i)^\Delta(t) = -H^i_x(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t))$

$$- \sum_{\substack{j=1 \\ j \neq i}}^N H^i_{u_j}(t, x_*(t), u_*^1(t), \ldots, u_*^N(t), (\psi^i)^\sigma(t)) \cdot \gamma_{*x}^j(t, x_a, x_*(t))$$

(γ_{*x}^j denotes the partial derivative of γ_*^j with respect to x);

4. $\psi^i(b) = 0$;

5. $H^i_{u^i}(t, x_*(t), u^1_*(t), \ldots, u^N_*(t), (\psi^i)^\sigma(t)) = 0$;

for all $i = 1, 2, \ldots, N$ and $t \in \mathbb{T}^k$.

Proof. Differently from Theorem 16, now the controls u^i depend explicitly on the state variable x. Suppose that the decision N-tuple $(u^1_*, \ldots, u^N_*)$ is a MPS Nash-equilibrium of Γ_N, and x_* is the associated state trajectory. Fix $i \in \{1, 2, \ldots, N\}$. The same reasoning as used in the proof of Theorem 16 permit to conclude that $(x_*(\cdot), u^i_*(\cdot))$ is a weak local minimizer of problem (7), where we suppose now that the controls are MPS. In the following we will prove that

$$\min_{\substack{\text{all admissible MPS controls}}} \widetilde{J}^i(u^i) = \min_{\substack{\text{all admissible OL controls}}} \widetilde{J}^i(u^i). \qquad (8)$$

An OL control can be considered as a MPS control, so it is clear that

$$\min_{\substack{\text{all admissible MPS controls}}} \widetilde{J}^i(u^i) \leq \min_{\substack{\text{all admissible OL controls}}} \widetilde{J}^i(u^i).$$

Since $u^i_*(t) = \gamma^i_*(t, x_a, x(t))$ is a MPS Nash-equilibrium control, by the assumption of admissibility, the equations $x^\Delta(t) = \widetilde{f}(x(t), \gamma^i_*(t, x_a, x(t)))$ and $x(a) = x_a$ define a unique trajectory x_*. With this trajectory we define now the OL control

$$v^i_*(t) := \gamma^i_*(t, x_a, x_*(t)).$$

Notice that

$$\widetilde{J}^i(v^i_*) = \min_{\substack{\text{all admissible MPS controls}}} \widetilde{J}^i(u^i),$$

and hence equality (8) holds. Observe also that $(x_*(\cdot), v^i_*(\cdot))$ is a weak local minimizer of problem (7). Applying Theorem 3 to $\mathcal{H}^i$, we conclude that there exists a delta differentiable function $\psi^i : \mathbb{T} \to \mathbb{R}^n$ on $\mathbb{T}^k$ such that

$$
\begin{aligned}
(\psi^i)^\Delta(t) &= -\mathcal{H}^i_x(t, x_*(t), v^i_*(t), (\psi^i)^\sigma(t)) \\
&= -H^i_x(t, x_*(t), u^1_*(t), \ldots, u^N_*(t), (\psi^i)^\sigma(t)) \\
&\quad - \sum_{\substack{j=1 \\ j \neq i}}^{N} H^i_{u_j}(t, x_*(t), u^1_*(t), \ldots, u^N_*(t), (\psi^i)^\sigma(t)) \cdot \gamma^j_{*x}(t, x_a, x_*(t)).
\end{aligned}
$$

The other conditions are obtained in a similar way. $\qquad \square$

Acknowledgments

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